

Comparison Theorems of Infinite Horizon Forward-Backward Stochastic Differential Equations

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Abstract

By the methods of probability and duality technique, we give some comparison theorems for the solutions of infinite horizon forward-backward stochastic differential equations.

Key words: Infinite horizon forward-backward stochastic differential equations, comparison theorem, duality technique.

1 Introduction

In this paper, we study the comparison theorems of the following infinite horizon fully coupled forward-backward stochastic differential equations (FBSDEs in short)

$$\begin{cases} dX_t = b(t, X_t, Y_t, Z_t) dt + \sigma(t, X_t, Y_t, Z_t) dB_t, \\ -dY_t = f(t, X_t, Y_t, Z_t) dt - Z_t dB_t, \\ X_0 = x, \quad Y_\infty = \Phi(X_\infty), \end{cases} \quad (1.1)$$

where $\{B_t\}_{t \geq 0}$ be a standard d -dimensional Brownian motion, defined on a probability space (Ω, \mathcal{F}, P) . Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the completion of the natural filtration generated by the Brownian motion $\{B_t\}_{t \geq 0}$, where \mathcal{F}_0 contains all P -null sets of \mathcal{F} , and $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. We introduce the

following spaces:

$$\begin{aligned} \mathcal{S}^2 &= \left\{ v_t, \ 0 \leq t \leq \infty; \ v_t \text{ is an } \mathcal{F}_t\text{-adapted process s.t. } \mathbf{E} \left[\sup_{0 \leq t \leq \infty} |v_t|^2 \right] < \infty. \right\}, \\ \mathcal{H}^2 &= \left\{ v_t, \ 0 \leq t \leq \infty; \ v_t \text{ is an } \mathcal{F}_t\text{-adapted process s.t. } \mathbf{E} \left[\int_0^\infty |v_t|^2 dt \right] < \infty. \right\}, \\ L^2 &= \left\{ \xi; \ \xi \text{ is an } \mathcal{F}_\infty\text{-measurable random variable s.t. } \mathbf{E} |\xi|^2 < \infty. \right\}, \end{aligned}$$

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and denote that $\mathcal{B}^3 = \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$.

It is well known that the Hamiltonian system arised in studying maximum principle of stochastic optimal control problems is a kind of fully coupled FBSDEs (see [14]). In mathematical finance, fully coupled FBSDEs can be encountered when one studied the problems of hedging options involved in a large investor in financial market (see [2, 6]). On the other hand, fully coupled FBSDEs can provide probabilistic interpretations for the solutions of a class of quasilinear parabolic and elliptic PDEs (see [13, 16, 19]) and have been investigated deeply. They were studied first by Antonelli [1] and Ma, Protter and Yong [9], Cvitanic and Ma [2], Duffie, Ma and Yong [5]. Consequently, FBSDEs were developed in Hu and Peng [7], Pardoux and Tang [13], Peng and Wu [14] and Yong [21].

In addition, infinite horizon FBSDEs play a role of studying the Black consol rate conjecture (see [5]). Peng and Shi [15] firstly proved the existence and uniqueness of solutions of infinite horizon FBSDEs, under some monotone assumptions, while the solutions are in a kind of infinite time integrable space and the infinite terminal values of the solutions must be decay. Recently Shi and Zhao [16] extended the results in [15] to a larger space where the solutions must not be decay in infinite horizon. Later Wu [18] gave an existence and uniqueness result for FBSDEs with stopping times under some similar monotone assumptions, but different Lipschitz conditions. When the stopping times take infinite value, the solutions in [18] are in the space \mathcal{B}^3 above and not necessarily decay. We will present the existence and uniqueness result of [18] in Section 2.

The comparison theorems of FBSDEs play an important role of the applications of FBSDEs. So many literatures discussed the comparison results for FBSDEs. By virtue of partial differential equation method, Cvitanic and Ma [2] obtained a comparison theorem of FBSDEs. They required the forward equation to be non-degenerate and the coefficients not to be random, i.e. in Markovian cases. Obviously their results can not treat the non-Markovian cases that the coefficients themselves are randomly distributed, which are often the fact in practice, e.g. in incomplete finance markets. Later for non-Markovian finite horizon FBSDEs, Wu [17] proved a comparison result at initial time, where the dimension of the forward system X is $n > 1$ and the one of the backward system Y is $m = 1$. Peng and Shi [15] gave a comparison theorem for the solutions X_t and Y_t of infinite horizon FBSDEs for any $t \in R^+$ with $n = m = 1$. Recently Wu and Xu [20] obtained some comparison results for FBSDEs on finite horizon for many situations, such as $m = 1, n > 1$ and $m > 1, n = 1$. Moreover, some results hold for the process Y_t on the whole interval of $t \in [0, T]$. However there is a little gap in [20]. For overcoming this gap, we import a condition (i.e. (H4), for details see the proof of Theorem 2 in this paper). We will extend all the results in [17, 20, 22] to infinite horizon FBSDEs in many different situations including multi-dimensional cases. In addition, we also extend the comparison theorem for one-dimension infinite horizon backward stochastic differential equations (BSDEs in short) given in [8] to multi-dimensional case. These comparison theorems are significant in the theories of stochastic optimal control, differential games, PDEs and mathematical finance and so on.

This paper is organized as follows: we give the preliminaries and assumptions in Section 2, all kinds of comparison theorems are proved in Section 3 by virtue of probability method.

2 Preliminaries: the existence and uniqueness to FBS-DEs

We consider FBSDE (1.1) under the following setting: for any $t \geq 0$, $(X, Y, Z) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m \times d}$,

$$\begin{aligned} b(t, X, Y, Z) &: \Omega \times [0, \infty] \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m \times d} \rightarrow \mathbf{R}^n, \\ \sigma(t, X, Y, Z) &: \Omega \times [0, \infty] \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m \times d} \rightarrow \mathbf{R}^{n \times d}, \\ f(t, X, Y, Z) &: \Omega \times [0, \infty] \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m \times d} \rightarrow \mathbf{R}^m, \\ \Phi(X) &: \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^m. \end{aligned}$$

We assume

(H1) $\forall (X, Y, Z) \in \mathbf{R}^{n+m+m \times d}$, $\Phi(X) \in \mathbf{L}^2$, b , σ , and f are progressively measurable and

$$\mathbf{E} \left(\int_0^\infty |b(s, 0, 0, 0)| ds \right)^2 + \mathbf{E} \left(\int_0^\infty |f(s, 0, 0, 0)| ds \right)^2 + \mathbf{E} \int_0^\infty |\sigma(s, 0, 0, 0)|^2 ds < \infty;$$

(H2) There exists a positive deterministic bounded function $u_1(t)$, such that $\forall t \geq 0$, $\forall (X_i, Y_i, Z_i) \in \mathbf{R}^{n+m+m \times d}$, $(i = 1, 2)$,

$$|l(t, X_1, Y_1, Z_1) - l(t, X_2, Y_2, Z_2)| \leq u_1(t) (|X_1 - X_2| + |Y_1 - Y_2| + |Z_1 - Z_2|),$$

$l = b, \sigma, f$, respectively, and $\int_0^\infty u_1(s) ds < \infty$, $\int_0^\infty u_1^2(s) ds < \infty$ there exists a positive constant C such that

$$|\Phi(X_1) - \Phi(X_2)| \leq C |X_1 - X_2|, \quad u_1 \leq C.$$

Given an $m \times n$ full rank matrix G and let us introduce the following notations

$$u = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, A(t, u) = \begin{pmatrix} -G^* f \\ Gb \\ G\sigma \end{pmatrix} (t, u),$$

where $G\sigma = (G\sigma_1 \dots G\sigma_d)$. We use the usual inner product and Euclidean norm in $\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^{m \times d}$. All the equalities and inequalities mentioned in this paper are in sense of $dt \times dp$ almost surely on $[0, \infty] \times \Omega$. In addition, throughout this paper, when we indicate two vectors ν^1, ν^2 satisfying $\nu^1 \geq \nu^2$, it means that we have $\nu^{1,j} \geq \nu^{2,j}$ for each of their components.

H3 $\forall u = (X, Y, Z), \bar{u} = (\bar{X}, \bar{Y}, \bar{Z}) \in \mathbf{R}^{n+m+m \times d}$, $\hat{X} = X - \bar{X}, \hat{Y} = Y - \bar{Y}, \hat{Z} = Z - \bar{Z}$,

$$\begin{cases} \langle A(t, u) - A(t, \bar{u}), u - \bar{u} \rangle \leq -\beta_1 u_1(t) |G\hat{X}|^2 - \beta_2 u_1(t) \left(|G^* \hat{Y}|^2 + |G^* \hat{Z}|^2 \right), \\ \langle \Phi(X) - \Phi(\bar{X}), G(X - \bar{X}) \rangle \geq \mu |G\hat{X}|^2, \end{cases}$$

or

$$\begin{cases} \langle A(t, u) - A(t, \bar{u}), u - \bar{u} \rangle \geq \beta_1 u_1(t) |G\hat{X}|^2 + \beta_2 u_1(t) \left(|G^*\hat{Y}|^2 + |G^*\hat{Z}|^2 \right), \\ \langle \Phi(X) - \Phi(\bar{X}), G(X - \bar{X}) \rangle \leq -\mu |G\hat{X}|^2, \end{cases}$$

where β_1, β_2 and μ , are given nonnegative constant with $\beta_1 + \beta_2 > 0, \mu + \beta_2 > 0$. Moreover, we have $\beta_1 > 0, \mu > 0$ (resp., $\beta_2 > 0$) when $m > n$ (resp., $n > m$). Particularly, when $m = n$, we set $G = I_n$. And we denote the transpose of matrix by the notation $*$.

H4 f is strictly monotone in X .

Proposition 1. *Assume (H1)-(H3) hold, then FBSDE (1.1) has a unique solution $(X(\cdot), Y(\cdot), Z(\cdot)) \in \mathcal{B}^3$.*

The proof can be seen in [18].

For convenience of notations, hereafter we set $d = 1$ in this paper. In order to prove multi-dimensional comparison theorems for FBSDE (1.1), we need the following so-called quasi-monotonicity conditions.

- (A1) For $\forall t \geq 0, \forall Y \in \mathbf{R}^m, Z \in \mathbf{R}^{m \times d}, b(t, \cdot, Y, Z) : \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies that $\forall X, \bar{X} \in \mathbf{R}^n$, if $X_j \leq \bar{X}_j$ ($1 \leq j \leq n$), while $X_s = \bar{X}_s$ ($s \neq j$), then $b^s(t, X, Y, Z) \leq b^s(t, \bar{X}, Y, Z)$. For $\forall Y \in \mathbf{R}^m, Z \in \mathbf{R}^m, \sigma(t, \cdot, Y, Z) : \Omega \times \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n$, satisfies that $\sigma^j(t, x) = \sigma^j(t, 0, 0, \dots, x_j, \dots, 0, Y, Z)$.
- (A2) For $\forall t \geq 0, \forall X \in \mathbf{R}^n, f_s^i(t, X, \cdot, \cdot) : \Omega \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ ($i=1, 2$) satisfies that $\forall Y, \bar{Y} \in \mathbf{R}^m, \forall Z, \bar{Z} \in \mathbf{R}^m$, if $Y_s = \bar{Y}_s, Z_s = \bar{Z}_s$ ($s \neq j$), $Y_j \geq \bar{Y}_j$, ($1 \leq j \leq n$), then $f_s^1(t, X, Y, Z) \geq f_s^2(t, X, \bar{Y}, \bar{Z})$.

3 Comparison theorem

3.1 1-dimensional cases of FBSDEs

We first give the following result for $n \geq 1, m \geq 1$, which plays a significant role in our following comparison results.

Theorem 2. *Let (X^1, Y^1, Z^1) and (X^2, Y^2, Z^2) be respectively the solutions of (1.1) corresponding to $X_0^1 = x_1 \in \mathbf{R}^n$ and $X_0^2 = x_2 \in \mathbf{R}^n$. Set*

$$\begin{aligned} \hat{U} &= U^1 - U^2 = (X^1, Y^1, Z^1) - (X^2, Y^2, Z^2) \\ &= (X^1 - X^2, Y^1 - Y^2, Z^1 - Z^2) = (\hat{X}, \hat{Y}, \hat{Z}). \end{aligned}$$

If (H1)-(H4) hold, then $\langle G\hat{X}_t, \hat{Y}_t \rangle \geq 0, \forall t \in R^+$. Moreover, $(\hat{X}_t, \hat{Y}_t, \hat{Z}_t) \mathbf{I}_{[\tau, \infty]}(t) \equiv 0$, where τ is an \mathcal{F}_t -stopping time defined by

$$\tau \doteq \inf \left\{ t \geq 0; \langle G\hat{X}_t, \hat{Y}_t \rangle = 0 \right\}.$$

Proof. For $\forall t \in \mathbf{R}^+, \forall T \in [t, \infty)$, applying Itô's formula to $\langle G\hat{X}, \hat{Y} \rangle$ on $[t, T]$, it follows that

$$\begin{aligned} & \mathbf{E}^{\mathcal{F}_t} \langle G\hat{X}_T, \hat{Y}_T \rangle - \langle G\hat{X}_t, \hat{Y}_t \rangle \\ &= \mathbf{E}^{\mathcal{F}_t} \int_t^T \langle A(s, U_s^1) - A(s, U_s^2), \hat{U}_s \rangle ds \\ &\leq -\beta_1 \mathbf{E}^{\mathcal{F}_t} \int_t^T u_1(s) |G\hat{X}_s|^2 ds - \beta_2 \mathbf{E}^{\mathcal{F}_t} \int_t^T u_1(s) \left(|G^* \hat{Y}_s|^2 + |G^* \hat{Z}_s|^2 \right) ds. \end{aligned}$$

Letting $T \rightarrow \infty$, and by (H3) it follows that

$$\begin{aligned} \langle G\hat{X}_t, \hat{Y}_t \rangle &\geq \mu \mathbf{E}^{\mathcal{F}_t} |G\hat{X}_\infty|^2 + \beta_1 \mathbf{E}^{\mathcal{F}_t} \int_t^\infty u_1(s) |G\hat{X}_s|^2 ds \\ &\quad + \beta_2 \mathbf{E}^{\mathcal{F}_t} \int_t^\infty u_1(s) \left(|G^* \hat{Y}_s|^2 + |G^* \hat{Z}_s|^2 \right) ds \\ &\geq 0. \end{aligned}$$

So we have $\langle G\hat{X}_t, \hat{Y}_t \rangle \geq 0, \forall t \geq 0$. It follows that, for the above given stopping time τ

$$\begin{aligned} & \mathbf{E} \left(\langle G\hat{X}_\infty, \hat{Y}_\infty \rangle - \langle G\hat{X}_\tau, \hat{Y}_\tau \rangle \right) \\ &= \mathbf{E} \int_\tau^\infty \langle A(s, U_s^1) - A(s, U_s^2), \hat{U}_s \rangle ds \\ &\leq -\beta_1 \mathbf{E} \int_\tau^\infty u_1(s) |G\hat{X}_s|^2 ds - \beta_2 \mathbf{E} \int_\tau^\infty u_1(s) \left(|G^* \hat{Y}_s|^2 + |G^* \hat{Z}_s|^2 \right) ds, \end{aligned}$$

which means that

$$\beta_1 \mathbf{E} \int_\tau^\infty u_1(s) |G\hat{X}_s|^2 ds + \beta_2 \mathbf{E} \int_\tau^\infty u_1(s) \left(|G^* \hat{Y}_s|^2 + |G^* \hat{Z}_s|^2 \right) ds \equiv 0.$$

When $m > n$, we have $\beta_1 > 0$, so $\hat{X}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0$. From Theorem 2.1 in [18], we get $\hat{Y}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0, \hat{Z}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0$.

When $m < n$, we consider the following FBSDEs

$$\begin{cases} d\hat{X}_t = \left[b_t^1 \hat{X}_t + b_t^2 \hat{Y}_t + b_t^3 \hat{Z}_t \right] dt + \left[\sigma_t^1 \hat{X}_t + \sigma_t^2 \hat{Y}_t + \sigma_t^3 \hat{Z}_t \right] dB_t, \\ -d\hat{Y}_t = \left[f_t^1 \hat{X}_t + f_t^2 \hat{Y}_t + f_t^3 \hat{Z}_t \right] dt - \hat{Z}_t dB_t, \\ \hat{X}_0 = X_0^1 - X_0^2, \quad \hat{Y}_\infty = \Phi(X_\infty^1) - \Phi(X_\infty^2), \end{cases}$$

where

$$\begin{aligned}
l_t^{1j} &= \begin{cases} \frac{l(t, X_t^{21}, \dots, X_t^{1j}, \dots, X_t^{1n}, Y_t^1, Z_t^1) - l(t, X_t^{21}, \dots, X_t^{2j}, \dots, X_t^{1n}, Y_t^1, Z_t^1)}{X_t^{1j} - X_t^{2j}}, & \text{if } X_t^{1j} \neq X_t^{2j}, \\ 0, & \text{otherwise,} \end{cases} \\
l_t^{2j} &= \begin{cases} \frac{l(t, X_t^2, Y_t^{21}, \dots, Y_t^{1j}, \dots, Y_t^{1m}, Z_t^1) - l(t, X_t^2, Y_t^{21}, \dots, Y_t^{2j}, \dots, Y_t^{1m}, Z_t^1)}{Y_t^{1j} - Y_t^{2j}}, & \text{if } Y_t^{1j} \neq Y_t^{2j}, \\ 0, & \text{otherwise,} \end{cases} \\
l_t^{3j} &= \begin{cases} \frac{l(t, X_t^2, Y_t^2, Z_t^{21}, \dots, Z_t^{1j}, \dots, Z_t^{1m}) - l(t, X_t^2, Y_t^2, Z_t^{21}, \dots, Z_t^{2j}, \dots, Z_t^{1m})}{Z_t^{1j} - Z_t^{2j}}, & \text{if } Z_t^{1j} \neq Z_t^{2j}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Here $l = b, \sigma, f$, respectively. And note that

$$\hat{X}_t = X_t^1 - X_t^2 = \begin{pmatrix} X_t^{11} - X_t^{21} \\ X_t^{12} - X_t^{22} \\ \vdots \\ X_t^{1n} - X_t^{2n} \end{pmatrix}.$$

\hat{Y}_t, \hat{Z}_t also have similar notation. Therefore, b_t^1 and σ_t^1 are $n \times n$ matrices. b_t^2, σ_t^2, b_t^3 and σ_t^3 are $n \times m$ matrices. f_t^1 is an $m \times n$ matrix. f_t^2 and f_t^3 are $m \times m$ matrices. When $m < n$, we have $\beta_2 > 0$, then $\hat{Y}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0, \hat{Z}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0$. Then we have

$$\hat{f}_t^1 \hat{X}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0,$$

From (H4) it follows that $\hat{X}_\tau = 0$, a.s.. Using the uniqueness of solutions of stochastic differential equations, it is obvious that $\hat{X}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0$. In all

$$\hat{X}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv \hat{Y}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv \hat{Z}_t \mathbf{I}_{[\tau, \infty]}(t) \equiv 0.$$

The proof is complete. \square

Now we first consider a simple case, that is, $n = m = 1$.

Theorem 3. Assume that $n = m = 1$ and (b, σ, f, Φ) satisfy (H1)-(H4). Let (X^j, Y^j, Z^j) ($j = 1, 2$) be the solutions of FBSDE (1.1) corresponding to $X_0^1 = x_1 \in \mathbf{R}$ and $X_0^2 = x_2 \in \mathbf{R}$, respectively. If $x_1 \geq x_2$, then $X_t^1 \geq X_t^2, Y_t^1 \geq Y_t^2, \forall t \in [0, \infty]$.

Proof. We set

$$\begin{aligned}
\tau_x &\doteq \inf \left\{ t \geq 0; \hat{X}_t = 0 \right\}, \\
\tau_y &\doteq \inf \left\{ t \geq 0; \hat{Y}_t = 0 \right\}.
\end{aligned}$$

Clearly $\tau \leq \tau_x, \tau \leq \tau_y$, where τ is defined in Theorem 2. From Theorem 2, it holds that $\langle \hat{X}_t, \hat{Y}_t \rangle \geq 0, \forall t \geq 0$, and

$$\begin{aligned}
\hat{X}_t &= 0, & \forall t \geq \tau_x, \\
\hat{Y}_t &= 0, & \forall t \geq \tau_y.
\end{aligned}$$

From the continuity of \hat{X}_t and \hat{Y}_t , it follows that $\hat{X}_t \geq 0$ and $\hat{Y}_t \geq 0$, $\forall t \geq 0$. The proof is completed. \square

3.2 Comparison theorem of multi-dimensional BSDEs on infinite horizon

We give a Gronwall's lemma which is vital to prove a comparison theorem for solutions of multi-dimensional infinite horizon BSDEs.

Lemma 4. *Let $\eta \in L^1(\Omega, \mathcal{F}, P)$, y_t is an adapted process, and $\varphi(s)$ is a deterministic function such that $\int_0^\infty \varphi(s) ds < \infty$. If $y_t \leq \mathbf{E}[\eta + \int_t^\infty \varphi_s y_s ds | \mathcal{F}_t]$, $\forall t \geq 0$, then, there exists a positive constant M such that $y_t \leq M \mathbf{E}[\eta | \mathcal{F}_t]$.*

Proof. From the assumptions, we can get

$$\mathbf{E} \left[y_s - \int_s^\infty \varphi_r y_r dr | \mathcal{F}_s \right] \leq \mathbf{E}[\eta | \mathcal{F}_s]. \quad (3.2.1)$$

Set $\phi_s = \varphi_s e^{-\int_s^\infty \varphi_r dr}$, it is easy to check that $(e^{-\int_s^\infty \varphi_r dr})' = \phi_s$. Multiplying ϕ_s on both sides of (3.2.1), it yields

$$\mathbf{E} \left[\phi_s y_s - \phi_s \int_s^\infty \varphi_r y_r dr | \mathcal{F}_s \right] \leq \mathbf{E}[\phi_s \eta | \mathcal{F}_s].$$

Integrating on $[t, \infty]$, it follows that

$$\int_t^\infty \mathbf{E} \left[\phi_s y_s - \phi_s \int_s^\infty \varphi_r y_r dr | \mathcal{F}_s \right] ds \leq \int_t^\infty \phi_s \mathbf{E}[\eta | \mathcal{F}_s] ds.$$

Taking conditional expectation under \mathcal{F}_t on both sides of the above inequality

$$\mathbf{E} \left[\int_t^\infty \mathbf{E} \left[\phi_s y_s - \phi_s \int_s^\infty \varphi_r y_r dr | \mathcal{F}_s \right] ds | \mathcal{F}_t \right] \leq \mathbf{E} \left[\int_t^\infty \phi_s \mathbf{E}[\eta | \mathcal{F}_s] ds | \mathcal{F}_t \right].$$

From the property of classical conditional expectation and Fubini theorem, it follows that

$$\mathbf{E} \left[\int_t^\infty \phi_s y_s ds - \int_t^\infty \phi_s \int_s^\infty \varphi_r y_r dr ds | \mathcal{F}_t \right] \leq \int_t^\infty \phi_s \mathbf{E}[\eta | \mathcal{F}_t] ds.$$

After the calculation of integration by parts, it is clear that

$$\mathbf{E} \left[\int_t^\infty \varphi_s y_s ds | \mathcal{F}_t \right] \leq \int_t^\infty \varphi_s e^{\int_t^s \varphi_r dr} \mathbf{E}[\eta | \mathcal{F}_t] ds. \quad (3.2.2)$$

Substituting (3.2.2) into the original inequality, the desired result is obtained. \square

Proposition 5. Consider the following backward stochastic differential equations on infinite horizon,

$$y_t = \xi + \int_t^\infty f(s, y_s, z_s) ds - \int_t^\infty z_s dW_s, \quad (3.2.3)$$

where $\xi \in L^2$, $f : \Omega \times \mathbf{R}^+ \times \mathbf{R}^m \times \mathbf{R}^{m \times d} \rightarrow \mathbf{R}^m$. We assume that

(h.1) for any $(y, z) \in \mathbf{R}^m \times \mathbf{R}^{m \times d}$, $f(\cdot, y, z)$ is an $\{\mathcal{F}_t\}$ -progressively measurable process such that $\mathbf{E} \left(\int_0^\infty f(s, 0, 0) ds \right)^2 < \infty$;

(h.2) f satisfies Lipschitz condition with Lipschitz function, that is, there exist two positive deterministic functions $\psi(t), \delta(t)$, such that $|f(t, y^1, z^1) - f(t, y^2, z^2)| \leq \psi(t)|y^1 - y^2| + \delta(t)|z^1 - z^2|$, $\forall (t, y^i, z^i) \in \mathbf{R}^+ \times \mathbf{R}^m \times \mathbf{R}^{m \times d}$, $(i = 1, 2)$, where $\int_0^\infty \psi(s) ds < \infty$, $\int_0^\infty \delta^2(s) ds < \infty$. If (h1) and (h2) hold, then (3.2.3) has a unique solution $(y_t, z_t) \in \mathcal{S}^2 \times \mathcal{H}^2$.

The proof is referred to [4].

Now, we are position to study a comparison theorem for multi-dimensional BSDEs on infinite horizon.

Theorem 6. Consider the following two BSDEs:

$$y_t^1 = \xi^1 + \int_t^\infty f^1(s, y_s^1, z_s^1) ds - \int_t^\infty z_s^1 dW_s \quad (3.2.4)$$

$$y_t^2 = \xi^2 + \int_t^\infty f^2(s, y_s^2, z_s^2) ds - \int_t^\infty z_s^2 dW_s \quad (3.2.5)$$

If $\xi^i \in L^2$, $i = 1, 2$, and $\xi^1 \geq \xi^2$, f^1, f^2 enjoy (h.1) and (h.2), $f^1(t, \cdot, \cdot)$ and $f^2(t, \cdot, \cdot)$ satisfy the assumption (A2), then $\forall t \geq 0$, $y_t^1 \geq y_t^2$, that is, $y_t^{1,j} \geq y_t^{2,j}$, $1 \leq j \leq m$.

Proof. From the above Proposition 5, (3.2.4) and (3.2.5) have a unique solution (y^1, z^1) and (y^2, z^2) , respectively. Set

$$\hat{y}_t = y_t^2 - y_t^1, \quad \hat{\xi} = \xi^2 - \xi^1, \quad \hat{f} = f^2 - f^1, \quad \hat{z} = z^2 - z^1.$$

Applying Tanaka-Itô's formula to $|\hat{y}_t^+|^2$ on $[t, \infty]$.

$$\begin{aligned} & \left| (\hat{y}_t^j)^+ \right|^2 + \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \sum_{i=1}^d |\hat{z}_s^{j,i}|^2 I_{\{y_s^1 < y_s^2\}} ds \\ &= 2\mathbf{E}^{\mathcal{F}_t} \int_t^\infty (\hat{y}_s^j)^+ [f^{2,j}(s, y_s^1, z_s^2) - f^{1,j}(s, y_s^2, z_s^2)] \mathbf{I}_{\{y_s^1 < y_s^2\}} ds - \mathbf{E}^{\mathcal{F}_t} \int_t^\infty (\hat{y}_s^j)^+ dL_s^j \end{aligned} \quad (3.2.6)$$

where L_s^j is local time for $y_s^{2,j} - y_s^{1,j}$ at 0 which claims $\mathbf{E}^{\mathcal{F}_t} \int_t^\infty (\hat{y}_s^j)^+ dL_s^j = 0$. Set

$$\begin{aligned}
\Delta &= f^{2,j}(s, y_s^2, z_s^2) - f^{1,j}(s, y_s^1, z_s^1) \\
&= f^{2,j}(s, y_s^{2,1}, \dots, y_s^{2,j}, \dots, y_s^{2,m}, z_s^{2,1}, \dots, z_s^{2,j}, \dots, z_s^{2,m}) \\
&\quad - f^{1,j}\left(s, y_s^{1,1} + (\hat{y}_s^1)^+, y_s^{1,2} + (\hat{y}_s^2)^+, \dots, y_s^{2,j}, \dots, y_s^{1,m} + (\hat{y}_s^m)^+, z_s^{1,1}, \dots, z_s^{2,j}, \dots, z_s^{1,m}\right) \\
&\quad + f^{1,j}\left(s, y_s^{1,1} + (\hat{y}_s^1)^+, y_s^{1,2} + (\hat{y}_s^2)^+, \dots, y_s^{2,j}, \dots, y_s^{1,m} + (\hat{y}_s^m)^+, z_s^{1,1}, \dots, z_s^{2,j}, \dots, z_s^{1,m}\right) \\
&\quad - f^{1,j}(s, y_s^{1,1}, \dots, y_s^{1,j}, \dots, y_s^{1,m}, z_s^{1,1}, \dots, z_s^{1,j}, \dots, z_s^{1,m}) \\
\Delta_1 &= f^{2,j}(s, y_s^{2,1}, \dots, y_s^{2,j}, \dots, y_s^{2,m}, z_s^{2,1}, \dots, z_s^{2,j}, \dots, z_s^{2,m}) \\
&\quad - f^{1,j}\left(s, y_s^{1,1} + (\hat{y}_s^1)^+, y_s^{1,2} + (\hat{y}_s^2)^+, \dots, y_s^{2,j}, \dots, y_s^{1,m} + (\hat{y}_s^m)^+, z_s^{1,1}, \dots, z_s^{2,j}, \dots, z_s^{1,m}\right) \\
\Delta_2 &= f^{1,j}\left(s, y_s^{1,1} + (\hat{y}_s^1)^+, y_s^{1,2} + (\hat{y}_s^2)^+, \dots, y_s^{2,j}, \dots, y_s^{1,m} + (\hat{y}_s^m)^+, z_s^{1,1}, \dots, z_s^{2,j}, \dots, z_s^{1,m}\right) \\
&\quad - f^{1,j}(s, y_s^{1,1}, \dots, y_s^{1,j}, \dots, y_s^{1,m}, z_s^{1,1}, \dots, z_s^{1,j}, \dots, z_s^{1,m}),
\end{aligned}$$

so $\Delta = \Delta_1 + \Delta_2$. Obviously, according to (A2), $\Delta_1 \leq 0$. From the Lipschitz condition of f^1 , we have the following inequality

$$\Delta_2 \leq \psi(s) \left(|(\hat{y}_s^1)^+| + \dots + |\hat{y}_s^j| + \dots + |(\hat{y}_s^m)^+| \right) + \delta(s) |\hat{z}_s^j|.$$

Immediately, from (3.2.6), it follows that

$$\begin{aligned}
& \left| (\hat{y}_t^j)^+ \right|^2 + \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \sum_{i=1}^d |\hat{z}_s^{j,i}|^2 \mathbf{I}_{\{y_s^1 < y_s^2\}} ds \\
& \leq 2\mathbf{E}^{\mathcal{F}_t} \int_t^\infty \left[(\hat{y}_s^j)^+ \psi(s) \left(|(\hat{y}_s^1)^+| + \dots + |\hat{y}_s^j| + \dots + |(\hat{y}_s^m)^+| \right) + (\hat{y}_s^j)^+ \delta(s) |\hat{z}_s^j| \mathbf{I}_{\{y_s^1 < y_s^2\}} \right] ds.
\end{aligned}$$

Set $\kappa(t) = \delta^2(t) \vee \psi(t)$, $t \geq 0$. Obviously, $\int_0^\infty \kappa(s) ds < \infty$. From the inequality $2ab \leq a^2 + b^2$, we get

$$\begin{aligned}
& \left| (\hat{y}_t^j)^+ \right|^2 + \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \sum_{i=1}^d |\hat{z}_s^{j,i}|^2 \mathbf{I}_{\{y_s^1 < y_s^2\}} ds \\
& \leq (m+d) \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \left| (\hat{y}_s^j)^+ \right|^2 \kappa(s) \mathbf{I}_{\{y_s^1 < y_s^2\}} ds \\
& \quad + \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \sum_{i=1}^m \left| (\hat{y}_s^i)^+ \right|^2 \kappa(s) \mathbf{I}_{\{y_s^1 < y_s^2\}} ds + \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \sum_{i=1}^d |\hat{z}_s^{j,i}|^2 \mathbf{I}_{\{y_s^1 < y_s^2\}} ds.
\end{aligned}$$

Finally, there is a sufficiently large constant $K' > 0$, such that

$$\left| (\hat{y}_t^j)^+ \right|^2 \leq K' \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \sum_{i=1}^m \left| (\hat{y}_s^i)^+ \right|^2 \kappa(s) \mathbf{I}_{\{y_s^1 < y_s^2\}} ds, \quad 1 \leq j \leq m.$$

Combining all of them, we have

$$\sum_{j=1}^m \left| (\hat{y}_t^j)^+ \right|^2 \leq m K' \mathbf{E}^{\mathcal{F}_t} \int_t^\infty \sum_{i=1}^m \left| (\hat{y}_s^i)^+ \right|^2 \kappa(s) \mathbf{I}_{\{y_s^1 < y_s^2\}} ds,$$

by Gronwall's Lemma 4, immediately, we can get

$$\sum_{j=1}^m \left| (\hat{y}_t^j)^+ \right|^2 = 0, \quad \forall t \in R^+,$$

that is $(\hat{y}_t^j)^+ = 0$. The proof is completed. \square

3.3 Multi-dimensional cases of FBSDEs

From now on, we consider the FBSDE (1.1) whose solution X or Y is multi-dimensional.

Theorem 7. Assume that $n = 1, m > 1$, and (b, σ, f, Φ) satisfy (H1)-(H4). Let (X^j, Y^j, Z^j) be the solutions of FBSDE (1.1) corresponding to $X_0^1 = x_1 \in \mathbf{R}$ and $X_0^2 = x_2 \in \mathbf{R}$, respectively. If $x_1 \geq x_2$, $f(t, X_t^1, Y, Z)$ and $f(t, X_t^2, Y, Z)$ satisfy (A2), then $X_t^1 \geq X_t^2, Y_t^1 \geq Y_t^2, \forall t \in [0, \infty]$.

Proof. Let $\hat{X}_t = X_t^1 - X_t^2, \hat{Y}_t = Y_t^1 - Y_t^2, \hat{Z}_t = Z_t^1 - Z_t^2$, we have

$$\begin{cases} d\hat{X}_t = \left(b_t^1 \hat{X}_t + b_t^2 \hat{Y}_t + b_t^3 \hat{Z}_t \right) dt + \left(\sigma_t^1 \hat{X}_t + \sigma_t^2 \hat{Y}_t + \sigma_t^3 \hat{Z}_t \right) dB_t, \\ -d\hat{Y}_t = \left(f_t^1 \hat{X}_t + f_t^2 \hat{Y}_t + f_t^3 \hat{Z}_t \right) dt - \hat{Z}_t dB_t, \\ \hat{X}_0 = x_1 - x_2, \quad \hat{Y}_\infty = \bar{\Phi} \cdot \hat{X}_\infty, \end{cases} \quad (3.3.1)$$

where $1 \leq j \leq m, l = b, \sigma, f$, respectively,

$$\begin{aligned} l_t^1 &= \begin{cases} \frac{l(t, X_t^1, Y_t^1, Z_t^1) - l(t, X_t^2, Y_t^1, Z_t^1)}{X_t^1 - X_t^2}, & \text{if } X_t^1 \neq X_t^2, \\ 0, & \text{otherwise,} \end{cases} \\ l_t^2 &= \begin{cases} \frac{l(t, X_t^2, Y_t^{21}, \dots, Y_t^{1j}, \dots, Y_t^{1m}, Z_t^1) - l(t, X_t^2, Y_t^{21}, \dots, Y_t^{2j}, \dots, Y_t^{1m}, Z_t^1)}{Y_t^{1j} - Y_t^{2j}}, & \text{if } Y_t^{1j} \neq Y_t^{2j}, \\ 0, & \text{otherwise,} \end{cases} \\ l_t^3 &= \begin{cases} \frac{l(t, X_t^2, Y_t^2, Z_t^{21}, \dots, Z_t^{1j}, \dots, Z_t^{1m}) - l(t, X_t^2, Y_t^2, Z_t^{21}, \dots, Z_t^{2j}, \dots, Z_t^{1m})}{Z_t^{1j} - Z_t^{2j}}, & \text{if } Z_t^{1j} \neq Z_t^{2j}, \\ 0, & \text{otherwise,} \end{cases} \\ \bar{\Phi} &= \begin{cases} \frac{\Phi(X_\infty^1) - \Phi(X_\infty^2)}{X_\infty^1 - X_\infty^2}, & \text{if } X_\infty^1 \neq X_\infty^2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now we define a stopping time $\tau \doteq \inf \{t \geq 0; \hat{X}_t = 0\}$. Consider the following FBSDE:

$$\begin{cases} d\hat{X}_t = \left(b_t^1 \hat{X}_t + b_t^2 \hat{Y}_t + b_t^3 \hat{Z}_t\right) dt + \left(\sigma_t^1 \hat{X}_t + \sigma_t^2 \hat{Y}_t + \sigma_t^3 \hat{Z}_t\right) dB_t, \\ -d\hat{Y}_t = \left(f_t^1 \hat{X}_t + f_t^2 \hat{Y}_t + f_t^3 \hat{Z}_t\right) dt - \hat{Z}_t dB_t, \\ \hat{X}_\tau = 0. \quad \hat{Y}_\infty = \bar{\Phi} \cdot \hat{X}_\infty. \end{cases} \quad (3.3.2)$$

Clearly, (3.3.2) has a unique solution on $[\tau, \infty]$, moreover, $(\hat{X}_t, \hat{Y}_t, \hat{Z}_t) \mathbf{I}_{[\tau, \infty]}(t) \equiv 0$. From the continuity of \hat{X}_t and $\hat{X}_0 = x_1 - x_2 \geq 0$, it follows that $\hat{X}_t \geq 0, \forall t \geq 0$. Since $f(t, X_t^1, Y, Z)$ and $f(t, X_t^2, Y, Z)$ satisfy (A2) and $\bar{\Phi} \geq 0$, from Theorem 6, it holds that $Y_t^1 \geq Y_t^2, t \geq 0$. The proof is completed. \square

Now we study the case for $n > 1, m = 1$.

Theorem 8. Assume that $n > 1, m = 1$, and (b, σ, f, Φ) satisfy (H1)-(H4). Let (X^j, Y^j, Z^j) be the solutions of FBSDE (1.1) corresponding to $X_0^1 = x_1 \in \mathbf{R}^n, X_0^2 = x_2 \in \mathbf{R}^n$, respectively. If $x_1 \geq x_2, b(t, \cdot, Y^1, Z^1)$ and $\sigma(t, \cdot, Y^1, Z^1)$ satisfy the assumption (A1), then $Y_t^1 \geq Y_t^2, \forall t \in [0, \infty]$.

Proof. Consider the following FBSDE:

$$\begin{cases} d\hat{X}_t = \left(b_t^1 \hat{X}_t + b_t^2 \hat{Y}_t + b_t^3 \hat{Z}_t\right) dt + \left(\sigma_t^1 \hat{X}_t + \sigma_t^2 \hat{Y}_t + \sigma_t^3 \hat{Z}_t\right) dB_t, \\ -d\hat{Y}_t = \left(f_t^1 \hat{X}_t + f_t^2 \hat{Y}_t + f_t^3 \hat{Z}_t\right) dt - \hat{Z}_t dB_t, \\ \hat{X}_0 = x_1 - x_2, \quad \hat{Y}_\infty = \bar{\Phi} \cdot \hat{X}_\infty. \end{cases} \quad (3.3.3)$$

where $1 \leq j \leq n, l = b, f, \sigma$, respectively,

$$\begin{aligned} l_t^{1,j} &= \begin{cases} \frac{l(t, X_t^{21}, \dots, X_t^{1j}, \dots, X_t^{1n}, Y_t^1, Z_t^1) - l(t, X_t^{21}, \dots, X_t^{2j}, \dots, X_t^{1n}, Y_t^1, Z_t^1)}{X_t^{1j} - X_t^{2j}}, & \text{if } X_{1j} \neq X_{2j}, \\ 0, & \text{otherwise,} \end{cases} \\ l_t^{2,j} &= \begin{cases} \frac{l(t, X_t^2, Y_t^1, Z_t^1) - l(t, X_t^2, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2}, & \text{if } Y_t^1 \neq Y_t^2, \\ 0, & \text{otherwise,} \end{cases} \\ l_t^{3,j} &= \begin{cases} \frac{l(t, X_t^2, Y_t^2, Z_t^1) - l(t, X_t^2, Y_t^2, Z_t^2)}{Z_t^1 - Z_t^2}, & \text{if } Z_t^1 \neq Z_t^2, \\ 0, & \text{otherwise,} \end{cases} \\ \bar{\Phi} &= \begin{cases} \frac{\Phi(X_\infty^{21}, \dots, X_\infty^{1j}, \dots, X_\infty^{1n}) - \Phi(X_\infty^{21}, \dots, X_\infty^{2j}, \dots, X_\infty^{1n})}{X_\infty^{1j} - X_\infty^{2j}}, & \text{if } X_\infty^{1j} \neq X_\infty^{2j}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then we use the duality technique to get the dual equations of the above FBSDE (3.3.3):

$$\begin{cases} dP_t = (f_t^2 P_t - (b_t^2)^* Q_t - (\sigma_t^2)^* K_t) dt + (f_t^3 P_t - (b_t^3)^* Q_t - (\sigma_t^3)^* K_t) dB_t, \\ -dQ_t = (-(f_t^1)^* P_t + (b_t^1)^* Q_t + (\sigma_t^1)^* K_t) dt - K_t dB_t, \\ P_0 = 1. \quad Q_\infty = -(\bar{\Phi})^* P_\infty. \end{cases} \quad (3.3.4)$$

It is easy to check that (3.3.4) satisfy (H1)-(H3), so there exists a unique triple (P_t, Q_t, K_t) in \mathcal{B}^3 . We define the stopping time $\tau = \inf \{t \geq 0; P_t = 0\}$. Consider the following FBSDE on $[\tau, \infty]$,

$$\begin{cases} d\bar{P}_t = (f_t^2 \bar{P}_t - (b_t^2)^* \bar{Q}_t - (\sigma_t^2)^* \bar{K}_t) dt + (f_t^3 \bar{P}_t - (b_t^3)^* \bar{Q}_t - (\sigma_t^3)^* \bar{K}_t) dB_t, \\ -d\bar{Q}_t = -(f_t^1)^* \bar{P}_t + (b_t^1)^* \bar{Q}_t + (\sigma_t^1)^* \bar{K}_t) dt - \bar{K}_t dB_t, \\ \bar{P}_\tau = 0, \quad \bar{Q}_\infty = -(\bar{\Phi})^* \bar{P}_\infty. \end{cases} \quad (3.3.5)$$

Clearly, (3.3.5) has a unique solution $(\bar{P}_t, \bar{Q}_t, \bar{K}_t)$ in \mathcal{B}^3 . Furthermore, $(\bar{P}_t, \bar{Q}_t, \bar{K}_t) \mathbf{I}_{[\tau, \infty]}(t) \equiv 0$. Let us set

$$\begin{aligned} P'_t &= \mathbf{I}_{[0, \tau]}(t) P_t + \mathbf{I}_{[\tau, \infty]}(t) \bar{P}_t, \\ Q'_t &= \mathbf{I}_{[0, \tau]}(t) Q_t + \mathbf{I}_{[\tau, \infty]}(t) \bar{Q}_t, \\ K'_t &= \mathbf{I}_{[0, \tau]}(t) K_t + \mathbf{I}_{[\tau, \infty]}(t) \bar{K}_t. \end{aligned}$$

So (P'_t, Q'_t, K'_t) is the solution of (3.3.4). According to $P_0 > 0$ and the definition of τ , we get $P_t \geq 0$. Applying Itô's formula to $\langle \hat{X}_t, Q_t \rangle + \hat{Y}_t P_t$ on $[0, \infty]$, we have

$$\langle \hat{X}_0, Q_0 \rangle - \hat{Y}_0 P_0 = 0.$$

So $\hat{Y}_0 = -\langle \hat{X}_0, Q_0 \rangle$. If we can prove $Q_0 \leq 0$, we can obtain $\hat{Y}_0 \geq 0$. Therefore, we study the following infinite horizon BSDE

$$\begin{cases} -dQ_t = -(f_t^1)^* P_t + (b_t^1)^* Q_t + (\sigma_t^1)^* K_t) dt - K_t dB_t, \\ Q_\infty = -(\bar{\Phi})^* \cdot P_\infty. \end{cases} \quad (3.3.6)$$

Thanks to (H3) and (H4), we deduce $(f_t^1)^* > 0$, $\bar{\Phi} \geq 0$. By (A1), it holds that

$$\frac{b^{1,j}(t, x^{2,1}, \dots, x^{1,j}, \dots, x^{1,n}, y^1, z^1) - b^{1,j}(t, x^{2,1}, \dots, x^{2,j}, \dots, x^{1,n}, y^1, z^1)}{x^{1,j} - x^{2,j}} \geq 0, \\ \sigma_{1,t}^{s,j} = 0, s \neq j.$$

We need the following another infinite horizon BSDE

$$\begin{cases} -d\tilde{Q}_t = ((b_t^1)^* \tilde{Q}_t + (\sigma_t^1)^* \tilde{K}_t) dt - \tilde{K}_t dB_t, \\ \tilde{Q}_\infty = 0. \end{cases} \quad (3.3.7)$$

Set $f^1(P_t, Q_t) = -(f_t^1)^* P_t + (b_t^1)^* Q_t + (\sigma_t^1)^* K_t$ and $f^2(\tilde{P}_t, \tilde{Q}_t) = (b_t^1)^* \tilde{Q}_t + (\sigma_t^1)^* \tilde{K}_t$. Obviously f^1 and f^2 satisfy (A2). From the above comparison Theorem 6, we obtain $Q_t \leq 0$, $\forall t \geq 0$. It implies that $\hat{Y}_0 \geq 0$, immediately.

Set $\tau_y = \inf \{t \geq 0; \hat{Y}_t = 0\}$. From Theorem 2, we have $\langle G\hat{X}_{\tau_y}, \hat{Y}_{\tau_y} \rangle \equiv 0$, moreover $(\hat{X}_t, \hat{Y}_t, \hat{Z}_t) \mathbf{I}_{[\tau_y, \infty]}(t) \equiv 0$, then we have $\hat{Y}_t \geq 0$, $\forall t \in [0, \infty]$. The proof is completed. \square

Lastly, we treat the case that the coefficients of FBSDE b , f , Φ , and initial value are all different in two FBSDEs. We can obtain a kind of comparison theorem which is only for Y_0 .

Theorem 9. Assume that $n > 1$, $m = 1$, and $(b^i, \sigma^i, f^i, \Phi^i)$, $i = 1, 2$, satisfy (H1)-(H4). Let (X^i, Y^i, Z^i) ($i = 1, 2$) be the solutions of FBSDE (1.1) corresponding to $X_0^1 = x_1 \in \mathbf{R}^n$, $X_0^2 = x_2 \in \mathbf{R}^n$, respectively. If $x_1 \geq x_2$, $b^1(t, X, Y, Z) \geq b^2(t, X, Y, Z)$, $f^1(t, X, Y, Z) \geq f^2(t, X, Y, Z)$, $\Phi^1(X) \geq \Phi^2(X)$, $\sigma^1 = \sigma^2 = \sigma$, besides $b^1(t, \cdot, Y^1, Z^1)$ and $\sigma^j(t, \cdot, Y^1, Z^1)$, $1 \leq j \leq n$, satisfy (A1), then $Y_0^1 \geq Y_0^2$.

Proof. By the similar arguments used in Theorem 8, we give the following FBSDE:

$$\begin{cases} d\hat{X}_t = \left(b_t^1 \hat{X}_t + b_t^2 \hat{Y}_t + b_t^3 \hat{Z}_t + b^1(t, X_t^2, Y_t^2, Z_t^2) - b^2(t, X_t^2, Y_t^2, Z_t^2) \right) dt \\ \quad + \left(\sigma_t^1 \hat{X}_t + \sigma_t^2 \hat{Y}_t + \sigma_t^3 \hat{Z}_t \right) dB_t, \\ -d\hat{Y}_t = \left(f_t^1 \hat{X}_t + f_t^2 \hat{Y}_t + f_t^3 \hat{Z}_t + f^1(t, X_t^2, Y_t^2, Z_t^2) - f^2(t, X_t^2, Y_t^2, Z_t^2) \right) dt - \hat{Z}_t dB_t, \\ \hat{X}_0 = x_1 - x_2, \quad \hat{Y}_\infty = \hat{\Phi} \cdot \hat{X}_\infty + \Phi^1(X_\infty^2) - \Phi^2(X_\infty^2), \end{cases}$$

where l_t^1, l_t^2, l_t^3 are defined in Theorem 8, and $l = b^1, f^1, \sigma^1$, while

$$\hat{\Phi} = \begin{cases} \frac{\Phi^1(X_t^{21}, \dots, X_t^{1j}, \dots, X_t^{1n}, Y_t^1, Z_t^1) - \Phi^1(t, X_t^{21}, \dots, X_t^{2j}, \dots, X_t^{1n}, Y_t^1, Z_t^1)}{X_{1j} - X_{2j}}, & \text{if } X_{1j} \neq X_{2j}, \\ 0, & \text{otherwise.} \end{cases}$$

Simultaneously, we introduce its dual equation:

$$\begin{cases} dP_t = \left(f_t^2 P_t - (b_t^2)^* Q_t - (\sigma_t^2)^* K_t \right) dt + \left(f_t^3 P_t - (b_t^3)^* Q_t - (\sigma_t^3)^* K_t \right) dB_t, \\ -dQ_t = \left(-(f_t^1)^* P_t + (b_t^1)^* Q_t + (\sigma_t^1)^* K_t \right) dt - K_t dB_t, \\ P_0 = 1, \quad Q_\infty = -(\hat{\Phi})^* \cdot P_\infty. \end{cases}$$

Applying Itô's formula to $\langle \hat{X}_t, Q_t \rangle + \hat{Y}_t P_t$ on $[0, \infty]$, we have

$$\begin{aligned} \hat{Y}_0 &= \mathbf{E} [\Phi^1(X_\infty^2) - \Phi^2(X_\infty^2)] P_\infty \\ &\quad - \mathbf{E} \int_0^\infty \langle b^1(t, X_t^2, Y_t^2, Z_t^2) - b^2(t, X_t^2, Y_t^2, Z_t^2), Q_t \rangle dt \\ &\quad + \mathbf{E} \int_0^\infty (f^1(t, X_t^2, Y_t^2, Z_t^2) - f^2(t, X_t^2, Y_t^2, Z_t^2)) P_t dt - \langle \hat{X}_0, Q_0 \rangle. \end{aligned}$$

From $P_t \geq 0$ and $Q_t \leq 0$, which can be proved similarly in Theorem 8, associated with $b^1(t, X_t^2, Y_t^2, Z_t^2) - b^2(t, X_t^2, Y_t^2, Z_t^2) \geq 0$, $f^1(t, X_t^2, Y_t^2, Z_t^2) - f^2(t, X_t^2, Y_t^2, Z_t^2) \geq 0$. we get that $\hat{Y}_0 \geq 0$, immediately. The proof is completed. \square

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